Stochastic fluid flow dynamics under location uncertainty

E. Memin

Fluminance

Introduction

Geophysical flow analysis

- Strong interest on the use of stochastic filters and ensemble methods for data assimilation and forecasting
- Particularly interresting to combine a partially known evolution law with noisy data

Difficulties

- Data and state variable evolution laws generally do not live at the same scales
  Example: in oceanography or meteorology, models at mesoscales and image data at submesoscales ⇒smoothing of the data and use of subgrid models
- Require stochastic version of the evolution law and a modeling of the dynamics errors

Introduction

Requirements

- Construct a large scale stochastic evolution model
- Enabling a clear interaction with finer scale time series of Eulerian data (such as images)
- An explicit (Eulerian) evolution law at least for the first moment

Goals

- Explore such an expression of Navier-Stokes equation with location uncertainties
- Extend this expression for simple Geophysical models
- Use such models for variational assimilation or ensemble filtering with image data

Location uncertainties

Principle

- Fluid particles displacement can be separated in two components: a smooth differentiable components **w**
- Uncertainty function uncorrelated in time but correlated in space $\boldsymbol{\sigma} d\mathbf{B}_t$
- Displacement:

$$d\mathbf{X}(\mathbf{x},t) = \mathbf{w}(\mathbf{X}(\mathbf{x},t),t)dt + \boldsymbol{\sigma}(\mathbf{X}(\mathbf{x},t),t)d\hat{\mathbf{B}}_t, \text{ with } \mathbf{X}(\mathbf{x},0) = \mathbf{x},$$

- Eulerian description of the velocity fields:

$$\mathbf{U}(\mathbf{x},t) = \mathbf{w}(\mathbf{x},t)dt + \boldsymbol{\sigma}(\mathbf{x},t)d\hat{\mathbf{B}}_t.$$

- **U** should be solution of Navier Stokes equation derived from Newton 2nd law
- $\Rightarrow \boldsymbol{\sigma} d\mathbf{B}_t$ differentiable in space

Noise term

Brownian motion field avatar

$$\hat{\mathbf{B}}_t^n(\mathbf{x}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{B}_t(\mathbf{x}_i)\varphi_\nu(\mathbf{x} - \mathbf{x}_i),$$

$\mathbf{B}_t(\mathbf{x}_i)$ independant d-dimensional (with $d = 2$ or 3) standard Brownian motions centered on fixed grid $S = \{\mathbf{x}_i, i = 1, \ldots, n\}$ and φ_ν d-dimentional Gaussian function of standard deviation ν

- zero mean Gaussian process with uncorrelated time increment
- Limiting spatial covariance such that

$$\mathbf{Q} = \lim_{n\to\infty} \mathbb{E}[\hat{\mathbf{B}}_t^n(\mathbf{x})\hat{\mathbf{B}}_t^{nT}(\mathbf{y})] = t\varphi_{\sqrt{2}\nu}(\mathbf{x} - \mathbf{y})\mathbb{I}_d,$$

- Symmetric positive definite operator of finite trace in $L^2(\mathbb{R}^d, \mu)$

Noise term

Brownian motion field avatar

- $\hat{B}^n$ is a Gaussian process and hence tends in law to a zero mean continuous process with the same limiting covariance $\mathbf{Q}$
- Limiting process denoted in a formal way as:

$$\hat{\mathbf{B}}_t(\mathbf{x}) \triangleq \mathbf{B}_t \star \varphi_\nu(\mathbf{x}) = \int_{\mathbb{R}^d} \mathbf{B}_t(\mathbf{x}')\varphi_\nu(\mathbf{x}-\mathbf{x}')d\mathbf{x}',$$

- Covariance trace of $\hat{\mathbf{B}}_t^n$ tends to the same bound as *tr* $\mathbf{Q}$

$$\begin{aligned} trQ &= \sum_{k=1}^{\infty} \mathbb{E}\langle \hat{\mathbf{B}}_t, e_k\rangle^2 = \mathbb{E}|\hat{\mathbf{B}}|_2^2 = \lim_{n\to\infty} \frac{td}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^d} \varphi_\nu^2(\mathbf{x}-\mathbf{x}_i)d\mathbf{x}, \\ &= td(4\pi\nu^2)^{-d/2} \end{aligned}$$

- The energy of the Brownian avatar hence depends on ν but not on the number of grid points

Noise term

White noise avatar and turbulent component

- Analogue of white noise avatar on Ω

$$\boldsymbol{\sigma}(\mathbf{x}, t) d\hat{\mathbf{B}}_t = \int_\Omega \boldsymbol{\sigma}_t(\mathbf{x}, \mathbf{y}) d\hat{\mathbf{B}}_t(\mathbf{y}) d\mathbf{y}.$$

- $\boldsymbol{\sigma}$ a linear bounded deterministic symmetric operator of $L^2(\mathbb{R}^d, \mu) \to L^2_0(\Omega)$ with null boundary condition.
- Assume to have a bounded norm (Hilbert-Schmidt operator): $\sum_{k \in \mathbb{N}} \|\boldsymbol{\sigma} e_k\|^2 < \infty$
- spatial covariance of the turbulent component $\boldsymbol{\sigma}_t d\hat{\mathbf{B}}_t$

$$\begin{aligned}
\mathbf{Q}(\mathbf{x}, s, \mathbf{y}, t) &= \lim_{n \to \infty} \frac{1}{n} dt \delta(t - s) \sum_{i=0}^{n} \boldsymbol{\sigma}(\mathbf{x}, \bullet, t) \star \varphi_\nu(\mathbf{x}_i) \boldsymbol{\sigma}(\bullet, \mathbf{y}, t) \star \varphi_\nu(\mathbf{x}_i) \\
&= dt \delta(t - s) \boldsymbol{\sigma}_{\varphi_\nu}(\mathbf{x}, t) \boldsymbol{\sigma}_{\varphi_\nu}(\mathbf{y}, t), \\
&\triangleq a(\mathbf{x}, \mathbf{y}, t) dt \delta(t - s),
\end{aligned}$$

- Temporal integration of diagonal terms $\Rightarrow$ quadratic variation process

Noise term

White noise avatar and Turbulent component

$$\boldsymbol{\sigma}(\mathbf{x},t)d\hat{\mathbf{B}}_t = \int_\Omega \boldsymbol{\sigma}_t(\mathbf{x},\mathbf{y})d\hat{\mathbf{B}}_t(\mathbf{y})d\mathbf{y}.$$

- Spatial covariance simplifies for homegeneous diffusion operator $\boldsymbol{\sigma}_t(\mathbf{x}-\mathbf{y})$

$$\begin{aligned}\mathbf{Q} &= \lim_{n\to\infty} dt\delta(t-s)\frac{1}{n}\sum_{i=0}^{n}\boldsymbol{\sigma}(\bullet,t)\star\varphi_\nu(\mathbf{x}-\mathbf{x}_i)\boldsymbol{\sigma}(\bullet,t)\star\varphi_\nu(\mathbf{y}-\mathbf{x}_i)\\ &= dt\delta(t-s)\boldsymbol{\sigma}(\bullet,t)\star\boldsymbol{\sigma}(\bullet,t)\star\varphi_{\sqrt{2}\nu}(\mathbf{x}-\mathbf{y}),\end{aligned}$$

- Quadratic variation process

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}[\boldsymbol{\sigma}(\bullet,t)\star\varphi_\nu(\mathbf{x}-\mathbf{x}_i)]^2dt =& dt\int_\Omega\left(\boldsymbol{\sigma}(\bullet,t)\star\varphi_\nu(\mathbf{x})\right)^2d\mathbf{x}\\ =& \boldsymbol{\Sigma}(t)dt\end{aligned}$$

Noise term

Kraichnan smooth model

$$d\mathbf{B}_t^{n\zeta}(\mathbf{x}) = \frac{1}{\sqrt{n}} \sum_i d\mathbf{B}_t(\mathbf{x}_i)\psi_\kappa^\gamma \star f^\zeta(\mathbf{x}-\mathbf{x}_i),$$

$$f^\zeta(\mathbf{x}) = C_\zeta \|\mathbf{x}\|^{\zeta/2} \quad 0 < \zeta < 2.$$

- Incompressible fluid $d\boldsymbol{\xi}_t^\zeta = \mathcal{P} \star d\mathbf{B}_t^\zeta$.
- Spectral correlation define as

$$\widehat{\mathbf{Q}}(\mathbf{k})_{ij} = |\mathbf{k}|^{-\zeta-d}(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2})(\widehat{\psi_\kappa^\gamma})^2.$$

- Quadratic variation process for a passband spectral cutoff ($\mathbb{1}_{[\kappa\gamma]}(\mathbf{k})$)

$$d < \boldsymbol{\xi}_t^\zeta(\mathbf{x}), \boldsymbol{\xi}_t^\zeta(\mathbf{x}) >_{ij} = \frac{dt C_\zeta}{(2\pi)^d} \frac{d-1}{d} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \zeta^{-1}(L^\zeta - \ell_D^\zeta)\delta_{ij}$$

Stochastic Reynolds transport theorem

Volumetric rate of change

Volumetric rate of change of a scalar process $q(\mathbf{x}, t)$ transported by a velocity field

$$d\mathbf{X}_t = \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\hat{\mathbf{B}}_t$$

is:

$$d\int_{\mathcal{V}(t)} q(\mathbf{x}, t)d\mathbf{x} =$$

$$\int_{\mathcal{V}(t)} dq_t + (\boldsymbol{\nabla}\cdot(q\mathbf{w}) - \sum_{i,j}\frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}q)_{|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0} + \|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\|^2 q)dt +$$

$$\boldsymbol{\nabla}\cdot(q\boldsymbol{\sigma}d\hat{\mathbf{B}}_t)d\mathbf{x}, \text{ with } a^{ij}(\mathbf{x}, \mathbf{x}, t) = \sum_k \boldsymbol{\sigma}_\nu^{ik}(\mathbf{x}, t)\boldsymbol{\sigma}_\nu^{kj}(\mathbf{x}, t),$$

Example for the smooth Kraichnan model:

$$d\int_{\mathcal{V}(t)} q(\mathbf{x}, t)d\mathbf{x} = \int_{\mathcal{V}(t)} [dq_t + (\boldsymbol{\nabla}\cdot(q\mathbf{w}) - \frac{1}{2}\gamma\Delta q)dt + \boldsymbol{\nabla}q^T d\boldsymbol{\xi}_t^\zeta]d\mathbf{x},$$

Stochastic Reynolds transport theorem

Mass conservation

Mass conservation constraint on the transported volume:

$$d\rho_t + \boldsymbol{\nabla}\cdot(\rho \mathbf{w})dt = \frac{1}{2}\Big(\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}\rho)_{|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0} + \|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\|^2\rho\Big)dt - \mathrm{D}iv(\rho\boldsymbol{\sigma}\, d\hat{\mathbf{B}}_t).$$

For a fluid with constant density, mass preservation implies

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\, d\hat{\mathbf{B}}_t) &= 0,\\
\boldsymbol{\nabla}\cdot\mathbf{w} &= 0,\\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\cdot\mathbf{a}) &= 0
\end{aligned}$$

Stochastic Reynolds transport theorem

Isochoric flows and isoneutral uncertainty

- Mass conservation constraint:

$$d_t\rho + \boldsymbol{\nabla}\rho\mathbf{w}dt - \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i x_j}(\rho a^{ij})dt = \boldsymbol{\nabla}\rho\sigma d\hat{\mathbf{B}}_t$$

- If the uncertainty $\boldsymbol{\sigma} d\hat{\mathbf{B}}_t$ lies on the isodensity surfaces:

$$\boldsymbol{\sigma}^{ij} = \delta^{ij} - \frac{\partial_{x_i}\rho(\mathbf{x})\partial_{x_j}\rho(\mathbf{y})}{\|\nabla\rho\|^2}\delta(\mathbf{x}-\mathbf{y}).$$

- Small slope assumption ($\sqrt{(\partial_x\rho)^2+(\partial_y\rho)^2} << \partial_z\rho$) $\Rightarrow$ diffusion tensor (and the quadratic variation) reads:

$$\mathbf{a}(\mathbf{x}) = \begin{pmatrix} 1 & 0 & \alpha_x(\mathbf{x}) \\ 0 & 1 & \alpha_y(\mathbf{x}) \\ \alpha_x(\mathbf{x}) & \alpha_y(\mathbf{x}) & |\boldsymbol{\alpha}(\mathbf{x})|^2 \end{pmatrix}, \quad \boldsymbol{\alpha} = -(\partial_x\rho/\partial_z\rho, \partial_x\rho/\partial_z\rho, 0)$$

Stochastic Reynolds transport theorem

Isochoric flows and isoneutral uncertainty

- $\nabla \cdot \boldsymbol{\sigma} = 0 \Rightarrow \boldsymbol{\alpha}$ constant along the depth axis $\partial_z \alpha_x = \partial_z \alpha_y = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{\alpha} = 0$.
- Mass conservation$\Rightarrow$ deterministic diffusion along the density tangent plane:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \rho^{T} \mathbf{w} = \frac{1}{2} \sum_{ij} \partial_{x_i} (a_{ij} \partial_{x_j} \rho)$$

- "Isoneutral" or "Isopycnal" diffusion for unresolved mesoscale eddies in large scales ocean dynamics simulations

Stochastic Reynolds transport theorem

Kraichnan model

- Mass conservation$\Rightarrow$ advection diffusion with multiplicative stochastic forcing

$$d_t\rho + \boldsymbol{\nabla}\rho^{T}\mathbf{w}dt - \gamma\frac{1}{2}\sum_{i,j}\Delta\rho dt = \boldsymbol{\nabla}\rho\sigma d\hat{\mathbf{B}}_t$$

- For mean-field dynamics ($\mathbf{w} = \mathbb{E}d\mathbf{X}_t$) mean density evolves as

$$\partial_t\bar{\rho} + \boldsymbol{\nabla}\bar{\rho}^{T}\mathbf{w} = \frac{1}{2}\gamma\Delta\bar{\rho}$$

- For isoneutral noise

$$\partial_t\rho + \boldsymbol{\nabla}\rho^{T}\mathbf{w} = \frac{1}{2}\gamma\Delta\rho$$

Conservation of momentum

Conservation of momentum

Newton second law

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \mathbf{w} d\mathbf{x} = F,$$

Considering stochastic conservation principle

$$d\int_{\mathcal{V}(t)} \rho(\mathbf{w}(\mathbf{x},t)dt + \boldsymbol{\sigma}(\mathbf{x},t)d\hat{\mathbf{B}}_t)d\mathbf{x} = \int_{\mathcal{V}(t)} F(\mathbf{x},t)d\mathbf{x}.$$

highly irregular $\Rightarrow$ interpreted in the sense of distribution

Conservation of momentum

Conservation of momentum

For every $h \in C_0^\infty(\mathbb{R}+)$:

$$\int h(t) \int_{\mathcal{V}(t)} F(\mathbf{x},t) d\mathbf{x} dt = -\int h'(t) \int_{\mathcal{V}(t)} \boldsymbol{\sigma}(\mathbf{x},t) d\hat{\mathbf{B}}_t d\mathbf{x} dt + \int h(t) d \int_{\mathcal{V}(t)} \rho \mathbf{w}(t,\mathbf{x}) d\mathbf{x} dt.$$

Since both side of this equation must have the same structure, the forces can be written as:

$$\int h(t) \int_{\mathcal{V}(t)} F(\mathbf{x},t) dt = -\int h'(t) \int_{\mathcal{V}(t)} \boldsymbol{\sigma}(t,\mathbf{x}) d\hat{\mathbf{B}}_t d\mathbf{x} + \int h(t) \int_{\mathcal{V}(t)} (f(t,\mathbf{x}) d\mathbf{x} dt + \boldsymbol{\theta}(t,\mathbf{x}) d\hat{\mathbf{B}}_t) d\mathbf{x}.$$

First terms of both equations identical and cancel out.

Conservation of momentum

Conservation of momentum

We have:

$$d\int_{\mathcal{V}(t)}\rho w_i d\mathbf{x} = \int_{\mathcal{V}(t)}(d(\rho w_i)_t + (\boldsymbol{\nabla}\cdot(\rho w_i\mathbf{w}) - \sum_{j,k}\frac{1}{2}\frac{\partial^2}{\partial x_j\partial x_k}(a^{jk}\rho w_i)_{|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0} +$$

$$\|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\|^2\rho w_i)dt + \mathrm{Div}(\rho w_i\boldsymbol{\sigma}d\hat{\mathbf{B}}_t))d\mathbf{x},\ \text{with } a^{ij}(\mathbf{x},t) = \sum_k \boldsymbol{\sigma}_\nu^{ik}(\mathbf{x},t)\boldsymbol{\sigma}_\nu^{kj}(\mathbf{x},t).$$

As for the forces:

- Body force and external forces

$$G = \int_{\mathcal{V}}\rho(g\,dt - 2\boldsymbol{\Omega}\times\mathbf{U})d\mathbf{x},$$

- Surface forces

$$S = \int_{\partial\mathcal{V}}\boldsymbol{\Sigma}dt\mathbf{n}ds = \int_{\mathcal{V}} -\boldsymbol{\nabla}(p dt + d\hat{p}) + \mu(\Delta\mathbf{U} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{U}),$$

Navier Stokes equations

Stochastic Navier Stokes equations

Incorporating (stochastic) mass preservation principle and the forces expression:

$$\begin{cases} \left(\left(\frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}\boldsymbol{\nabla}^T\mathbf{w}\right)\rho - \frac{1}{2}\sum_{i,j} a_{ij}\rho\frac{\partial^2 \mathbf{w}}{\partial x_i \partial x_j} - \sum_{i,j}\frac{\partial(a_{ij}\rho)}{\partial x_j}\Big|_{\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0}\frac{\partial \mathbf{w}}{\partial x_i}\right)dt + \\ \quad \mathbf{w}\boldsymbol{\nabla}^T\rho\boldsymbol{\sigma}d\hat{\mathbf{B}}_t = \left(\rho g - 2\rho\boldsymbol{\Omega}\times\mathbf{w} - \boldsymbol{\nabla}p + \mu(\Delta\mathbf{w} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{w})\right)dt - \\ \quad \boldsymbol{\nabla}d\hat{p} - 2\rho\boldsymbol{\Omega}\times(\boldsymbol{\sigma}d\hat{\mathbf{B}}_t) + \mu(\Delta(\boldsymbol{\sigma}d\hat{\mathbf{B}}_t) + \frac{1}{3}\boldsymbol{\nabla}(\mathrm{D}iv(\boldsymbol{\sigma}d\hat{\mathbf{B}}_t))), \\ d\rho_t + \left(\boldsymbol{\nabla}\cdot(\rho\mathbf{w}) - \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\rho)_{|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0} + \|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\|^2\rho\right)dt = \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}d\hat{\mathbf{B}}_t). \end{cases}$$

Navier Stokes equations

Stochastic Navier Stokes equations

Equating slow terms and highly oscillating terms:

$$\begin{cases} (\frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}\boldsymbol{\nabla}^T\mathbf{w})\rho - \frac{1}{2}\sum_{i,j} a_{ij}\rho\frac{\partial^2 \mathbf{w}}{\partial x_i \partial x_j} - \sum_{i,j}\frac{\partial(a_{ij}\rho)}{\partial x_j}|_{\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0}\frac{\partial \mathbf{w}}{\partial x_i} = \\ \rho g - 2\rho\boldsymbol{\Omega}\times\mathbf{w} - \boldsymbol{\nabla}p + \mu(\Delta\mathbf{w} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{w})), \\ \boldsymbol{\nabla}d\hat{p}_t = -\mathbf{w}\boldsymbol{\nabla}^T\rho\boldsymbol{\sigma}d\hat{\mathbf{B}}_t - 2\rho\boldsymbol{\Omega}\times(\boldsymbol{\sigma}d\hat{\mathbf{B}}_t) + \mu(\Delta(\boldsymbol{\sigma}d\hat{\mathbf{B}}_t) + \\ \frac{1}{3}\boldsymbol{\nabla}(\mathrm{D}iv(\boldsymbol{\sigma}d\hat{B}_t))), \\ d\rho_t + \boldsymbol{\nabla}\cdot(\rho\mathbf{w}) - \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\rho)_{|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=0} + \|\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\|^2\rho) = \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}d\hat{\mathbf{B}}_t), \end{cases}$$

Navier Stokes equations

Stochastic Navier Stokes equations for the smooth Kraichnan model

$$\forall \mathbf{x} \in \Omega \text{ , } t \in]0, T]$$

$$\begin{cases} (\frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}\boldsymbol{\nabla}^T\mathbf{w} - \gamma\frac{1}{2}\Delta\mathbf{w})\rho = \rho g - 2\rho\boldsymbol{\Omega} \times \mathbf{w} - \boldsymbol{\nabla} p + \mu\Delta\mathbf{w} \\ \boldsymbol{\nabla} d\hat{p}_t = -\rho(\mathbf{w}\boldsymbol{\nabla}^T)d\boldsymbol{\xi}_t + 2\rho\boldsymbol{\Omega} \times d\boldsymbol{\xi}_t + \mu\Delta d\boldsymbol{\xi}_t, \\ \boldsymbol{\nabla} \cdot \mathbf{w} = 0, \end{cases}$$

with boundary and initial conditions:

$$\begin{cases} \mathbf{w} \cdot \mathbf{n} = 0 \;\text{ on }\; \partial\Omega, \; \mathrm{t} \in]0, \mathrm{T}], \\ d\boldsymbol{\xi}_t = 0 \;\text{ on }\; \partial\Omega, \; \mathrm{t} \in]0, \mathrm{T}], \\ \mathbf{w}_{|_{t=0}} = \mathbf{w}_o \;\text{ in }\; \Omega. \end{cases}$$

Navier Stokes equations

Stochastic Navier Stokes equations for incompressible and divergence free general turbulent model

$$\begin{cases} (\frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}\boldsymbol{\nabla}^T\mathbf{w} - \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i x_j}(a^{ij}\mathbf{w}))\rho = \rho g - 2\rho\boldsymbol{\Omega}\times\mathbf{w} - \boldsymbol{\nabla} p + \mu\Delta\mathbf{w} \\ \boldsymbol{\nabla} d\hat{p}_t = -\rho(\mathbf{w}\boldsymbol{\nabla}^T)\boldsymbol{\sigma} d\hat{\mathbf{B}}_t + 2\rho\boldsymbol{\Omega}\times\boldsymbol{\sigma} d\hat{\mathbf{B}}_t + \mu\Delta\boldsymbol{\sigma} d\hat{\mathbf{B}}_t, \\ \boldsymbol{\nabla}\cdot\mathbf{w} = 0, \\ \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\cdot\mathbf{a}) = 0 \end{cases}$$

with boundary and initial conditions:

$$\begin{cases} \mathbf{w}\cdot\mathbf{n} = 0 \ \text{ on } \ \partial\Omega, \ \mathrm{t}\in]0,\mathrm{T}], \\ \boldsymbol{\sigma} = 0 \ \text{ on } \ \partial\Omega, \ \mathrm{t}\in]0,\mathrm{T}], \\ \mathbf{w}_{|_{t=0}} = \mathbf{w}_o \ \text{ in } \ \Omega. \end{cases}$$

Navier Stokes equations

Subgrid model

- Energy dissipating

$$\int_\Omega \mathbf{w}^T \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}\mathbf{w})d\mathbf{x} = -\int_\Omega \|\nabla \mathbf{w}\|_{\mathbf{a}}^2 d\mathbf{x}$$

Navier Stokes equations

Link to Smagorinsky model

- Smagorinsky model $\nabla \cdot (c\|S\|S)$, $\|S\|^2 = \frac{1}{2}\sum_{ij}(\partial_{x_i} w^j + \partial_{x_j} w^i)^2$
- Taking $\mathbf{a} = c\|S\|\mathbb{I} \Rightarrow$
  $\sum_{ij} \partial_{x_i}\partial_{x_j} a_{ij}\mathbf{w} = 2\sum_j \partial_{x_j}\|S\|\partial_{x_j} w^k + \|S\|\Delta w^k + \Delta\|S\| w^k$
  Complemented by $c\sum_j \partial_{x_j}(\|S\|)\partial_{x_k} w^j - \Delta\|S\| w^k$ provides the standard trace free Smagorinsky subgrid stress
- The complementary term may be rewritten as

$$\underbrace{2\,\partial_{x_k}\sum_j \partial_{x_j}(\|S\|)w^j}_{(1)} - \underbrace{2\sum_j \partial_{x_j}\partial_{x_k}(\|S\|)w^j - \Delta\|S\| w^k}_{(2)}$$

- (1) gradient term $\Rightarrow$ compensated by a modified pressure
- Assuming (2) cancels we recover the Smagorinsky model

$\Rightarrow$ $\|S\|$ very smooth (respect $\nabla \cdot \nabla a = 0$), or w leaving on a manifold defined by the kernel of the Hessian of $\|S\|$

Results

Simulation Navier-Stokes drift

$$(\frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}\boldsymbol{\nabla}^T\mathbf{w} - \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i x_j}(a^{ij}\mathbf{w}))\rho = -\boldsymbol{\nabla} p + \mu\Delta\mathbf{w}$$

$$\boldsymbol{\nabla}\cdot\mathbf{w} = 0, \ \ \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\cdot\mathbf{a}) = 0, \ \ \text{periodic boundary conditions}$$

- Eliminating the pressure with Leray projector $\mathbb{P}$ computed on a divergence free wavelet basis

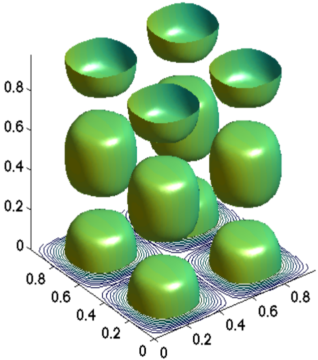
$$\frac{\partial \mathbf{w}}{\partial t} - \nu\Delta\mathbf{w} = \mathbb{P}[\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}\mathbf{w}) - \mathbf{w}\boldsymbol{\nabla}^T\mathbf{w}],$$

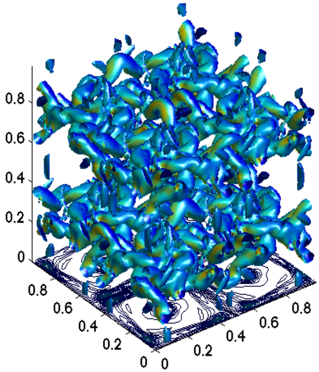
- Implicit Euler scheme expressed on $\mathbf{w}(t,\mathbf{x}) = \sum d_{j,k}(t)\ \Psi^{div}_{j,k}(\mathbf{x})$

$$(I - \nu\delta t\Delta)\mathbf{w}^{n+1} = \mathbf{w}^n - \delta t\mathbb{P}[\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}\mathbf{w}^n) - \mathbf{w}^n\boldsymbol{\nabla}^T\mathbf{w}^n].$$

- Variance tensor $a_{ij}(x)$ fixed from spatial or temporal variance in a local neighborhood

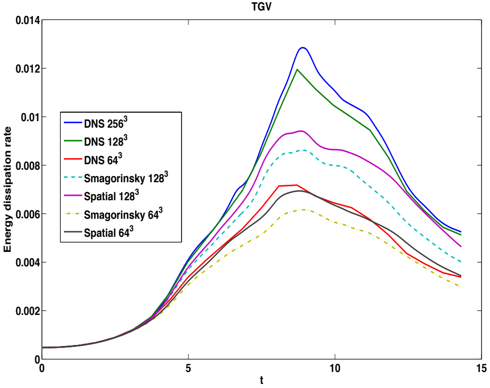
Results Green-Taylor





Green-Taylor vortex initial configuration isovalue subsequent time

Results Green-Taylor

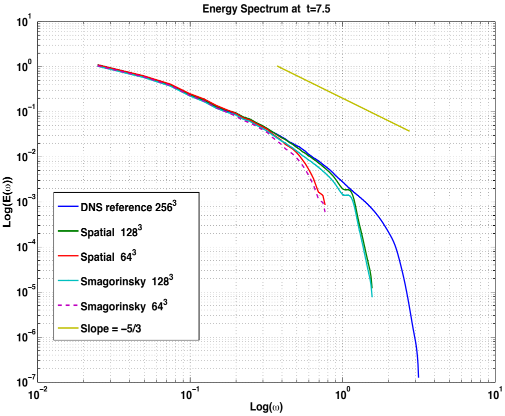


Evolution of the dimensionless energy dissipation rate as a function of the dimensionless time.

Results Crow instability

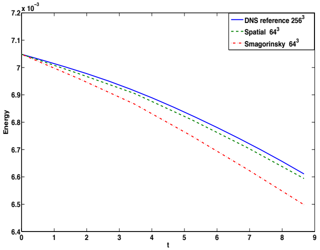
Crow instability vortex Smagorinsky and spatial variance

Results Crow instability

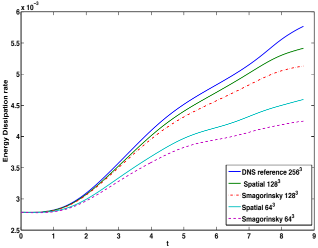


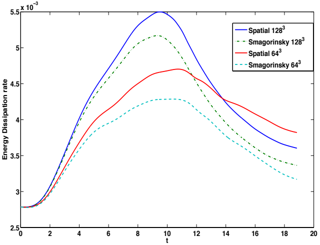
Energy spectrum.

Results Crow instability



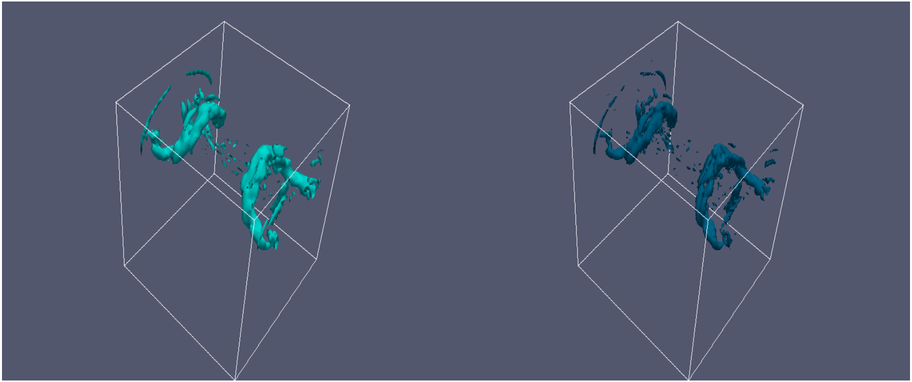
Energy





Energy dissipation rate

Results Crow instability



Smagorinsky (64^3, $t = 11$) Spatial Cov. (64^3, $t = 11$)

Navier Stokes equations under uncertainty

Shallow water under uncertainty

- General framework to derive large scale geophysical models
- Example Shallow Water

$$
\begin{aligned}
&(\frac{\partial \mathbf{w}^h}{\partial t} + \mathbf{w}^h \boldsymbol{\nabla}^T \mathbf{w}^h - \frac{1}{2} \sum_{(i,j)^h} \partial_{x_i} \partial_{x_j} (a_{ij} \mathbf{w}^h) \rho = -g \rho \boldsymbol{\nabla} h_u, \\
&d_t h + (\boldsymbol{\nabla} \cdot (h \mathbf{w}^h) - \frac{1}{2} \sum_{i,j} \partial_{x_i} \partial_{x_j} (a_{ij} h)) dt + \boldsymbol{\nabla} h (\boldsymbol{\sigma} d \hat{\mathbf{B}}_t)^h = 0, \\
&\boldsymbol{\nabla}^h d\hat{p} = -\rho (\mathbf{w}^h \boldsymbol{\nabla}^T)(\boldsymbol{\sigma} d \hat{\mathbf{B}}_t)^h, \\
&\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^h = 0, \\
&\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \cdot \mathbf{a}^h) = 0.
\end{aligned}
$$

Navier Stokes equations under uncertainty

Shallow water under uncertainty

- General framework to derive large scale geophysical models
- Example Shallow Water (free surface expectation or uncertainty on iso-height surface)

$$(\frac{\partial \mathbf{w}^h}{\partial t} + \mathbf{w}^h \boldsymbol{\nabla}^T \mathbf{w}^h - \frac{1}{2} \sum_{(i,j)^h} \partial_{x_i} \partial_{x_j} (a_{ij} \mathbf{w}^h))\rho = -g\rho \boldsymbol{\nabla} \bar{h}_u,$$

$$\frac{\partial \bar{h}}{\partial t} + \boldsymbol{\nabla} \cdot (\bar{h} \mathbf{w}^h) - \frac{1}{2} \sum_{(i,j)^h} \partial_{x_i} \partial_{x_j} (a_{ij} \bar{h}) = 0,$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \cdot \mathbf{a}^h) = 0.$$

Conclusion

- Derivation of a stochastic expression of Navier-Stokes
- Identification of the mean evolution equation
- Subgrid term related to the variance of the random turbulent term
- Identification of this variance through image data ?
- No model for the variance or covariance evolution

Navier Stokes equations under uncertainty

Perspectives

- Derivation of geophysical models under uncertainty
- Ertel's theorem ?
- Surface quasi-geostrophic ?
- Oceanic model ?
- Data assimilation from small scales observations